A deterministic algorithm for station keeping with a fixed-thruster geostationary spacecraft

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1 Abstract

Maintaining the precise geostationary position of a satellite during its operational lifetime is a critical challenge in satellite mission design. Traditional approaches to station-keeping rely on mechanisms such as thruster orientation or thrust-pointing strategies to maintain orbital stability and angular momentum management. However, these solutions add design complexity, mechanical vulnerabilities, and increased costs and mass.

This work presents a novel deterministic algorithm tailored for fixed-thrust geostationary satellites using electric thrusters. The proposed method provides a deterministic algorithm for computing station-keeping maneuvers and a maneuver planning scheme. The algorithm works to accommodate for all combinations of position or direction of thruster and is addressing orbit control and angular management at the same time. This approach allows building maneuver planning to achieve similar orbit management and angular momentum management performance compared to traditional methods while simplifying satellite design.

We will present our new maneuver scheme named hopping scheme. This scheme allows compensating for perturbation by hopping back and forth in eccentricity and angular momentum to achieve no intersection of burn during the station keeping cycle. With this, we manage to stay in $12 * 10^{-3}$ deg geostationary box size and below 15 Nms over the station keeping cycle, assuming a 1000 kg spacecraft. This hopping scheme is flexible, accommodating a large type of requirement via variation of its parameters.

This paper aims to provide readers with an understanding of the algorithm's development, its practical applications, and performance metrics. By leveraging this approach, mission designers can simplify satellite architectures while maintaining mission reliability.

2 Introduction

Geostationary satellites are indispensable for telecommunications, weather forecasting, and Earth observation due to their ability to maintain a fixed position relative to the Earth's surface. However, sustaining this position requires constant station-keeping maneuvers to counteract perturbations caused by 3-body perturbation from the Moon and the Sun, gravitational irregularities, solar radiation pressure, and other environmental forces. Traditional station-keeping solutions rely heavily on adjustable thrusters and thrust-pointing mechanisms, which introduce significant complexity, cost, and mass, in addition to susceptibility to mechanical failures over time.

This research introduces a novel deterministic algorithm for fixed-thrust geostationary satellites that eliminates the need for adjustable thrusters while ensuring precise station-keeping performance. Although developed with electric propulsion in mind, it can be technically used for all kinds of propulsion. The algorithm is also design-agnostic, meaning it can be implemented on a wide range of satellite platforms, making it particularly appealing for smallsat missions where simplicity and cost-efficiency are paramount.

The maneuver determination algorithm can be summarized as follows: The inputs to the algorithm are spacecraft design and required changes in orbital elements and angular momentum. The design inputs build a system of equations based on thrusters direction and thrusters position with respect to the center of mass. This system of equations is completed by the required performance of the maneuver. By integrating the angular momentum into the system of equations, we make a fully determined system that enables us to have a deterministic solution and bypass thrusters orientation or pointing mechanism to take care of the angular momentum with requirement definition without any sacrifice in performance.

This algorithm is to be seen as a foundational building block in the maneuver planning scheme. It needs to be incorporated into a larger algorithm to be able to produce a full maneuvering scheme. An optimization layer could be added to insure fuel efficiency for the mission's lifetime, or a validation step could be added to ensure maneuvers avoid operational constraints, such as avoiding burn during an eclipse.

To demonstrate the algorithm's effectiveness, a new maneuver scheme, named the "hopping maneuver," has been developed. This strategy allows the satellite to remain within a geostationary box of 1.2×10^{-3} deg while keeping angular momentum below a 15 Nms threshold during the station-keeping cycle. The "hop" maneuver works by alternately adjusting eccentricity and angular momentum, balancing the effects of orbital perturbations. A typical cycle consists of two maneuvers: a "hop forth" and a "hop back," both of which occur within a single day, while ensuring that maneuvers do not overlap or interfere with each other. The size of the hop is a tunable parameter that, together with the spacecraft design, is customized to meet specific mission requirements, such as fuel constraints or mission lifetime.

The deterministic algorithm for fixed-thrust station-keeping offers a compelling alternative to traditional thruster pointing or thruster orientation-based approaches. By simplifying satellite design, this approach aims to transform station-keeping strategies, particularly for smallsat missions where cost, mass, and operational simplicity are critical.

At the SmallSat 2025 Conference, attendees will gain insights into the development of this algorithm, its practical applications, and its potential to revolutionize station-keeping operations in the context of smallsat constellations. The algorithm's ability to integrate seamlessly with existing and future propulsion technologies, as well as its scalability across diverse satellite platforms, makes it a compelling solution for a wide range of space mission scenarios.

3 Introduction of the physical model

The coordinate system used in this algorithm is inspired by [5], which is based on the classical orbital elements: semimajor axis (a), eccentricity (e), inclination (i), right ascension of the ascending node (Ω), argument of perigee (ω), and true anomaly (ν). These elements describe the elliptic orbit of unperturbed motion, that is, an orbit only influenced by a spherically symmetric gravitational field around a body, Earth in this case.

The orbital elements are defined in relation to the inertial system called ECI (Earth-centered inertial). The origin is in the center of mass of Earth and is fixed with respect to the stars. The x-y-plane is in the equatorial plane, where x points towards the sun at vernal equinox and y is 90° east from X. The axe Z is pointing north, normal to the equatorial plane.

The semimajor axis is defined by the shortest distance to the center of Earth, perigee r_P, and the longest, apogee r_A, $a = (r_A + r_P)/2$. The eccentricity describes how elongated the ellipse is, where e = 0 is a circular orbit. It is defined by $e = \frac{r_A}{a} - 1$ and satisfies the inequality $0 \leq e < 1$. The angle between the orbital plane and the equator plane is the inclination of the orbit. For a geostationary orbit, it is close to 0. The orientation of the inclined orbit is determined by the right ascension of the ascending node (RAAN), which is the angle between the ECI x-axis and the ascending node. Furthermore, the argument of perigee is the angle between the ascending node and the perigee. Lastly, the true anomaly determines the current position of the spacecraft, by the angle around the origin, starting from perigee.

In this algorithm, eccentricity and inclination are treated as vectors. The three-dimensional inclination vector is a normalized vector normal to the orbital plane. The projection of this vector onto the ECI x-y-plane is,

$$i = (\sin(i)\sin(\Omega), -\sin(i)\cos(\Omega)) \tag{1}$$

and for small inclinations it can be linearized by $\sin(i) \approx i$, to

$$i = i(\sin(\Omega), -\cos(\Omega)). \tag{2}$$

The eccentricity can be defined in the same manner,

$$e = e(\cos(\Omega + \omega), \sin(\Omega + \omega)). \tag{3}$$

For a geostationary orbit of unperturbed motion, the spacecraft would be at rest with respect to Earth. When introducing perturbations, the motion relative to this ideal position can be described by the eccentricity and inclination vector and drift. Drift is the mean longitude drift rate, defined by

$$D = -1.5\frac{\delta a}{A}, \tag{4}$$

where A is the semimajor axis of the unperturbed geostationary orbit, 42164.2 km, and δa is deviation from that, $\delta a = a - A$. D is a dimensionless parameter. To get the longitude drift in degrees/day, D has to be multiplied by 361 deg/day.

The local coordinate system of the spacecraft used in this algorithm is RTO: radial, tangential, and orthogonal, shown in 1. This system rotates in ECI as the spacecraft rotates around Earth. It is defined in ECI by the time dependent sideral angle, s_m, of the spacecraft,

$$\begin{aligned} r &= (\cos(s_m, \sin(s_m), 0) && (5) \\ t &= (-\sin(s_m, \cos(s_m), 0) && (6) \\ o &= (0, 0, 1) && (7) \\ & && (8) \end{aligned}$$

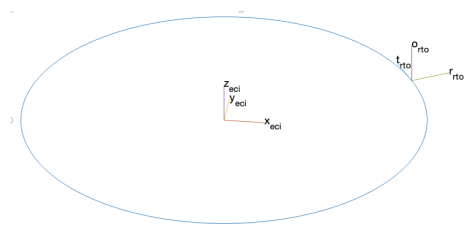

Figure 1: Definition of the RTO orbit frame with respect to the ECI orbit frame

4 Algorithm

4.1 Overall Overview

The general scheme of maneuver computations goes in three steps. The first step is to determine how much change in orbital element and angular momentum the maneuver should achieve. The amount is determined such that the maneuver will compensate the various perturbation undergone by the spacecraft, correct for potential error in the position of the spacecraft or compensate for buildup in angular momentum. All of these parameters may be pondered by strategies to optimize fuel over a longer period of time. Generally, it is done on ground via orbit porpagation or perturbation estimation.

The second step is the determination of the maneuver that can achieve the required performance defined by the previous step. It is using a deterministic algorithm which takes into account the design to produce the correct maneuver. This step produces when and for how long each thruster should be fired.

The third step is the post processing step, where the output of the second step is adapted to the need and requirement of the missions. For example, it may check if burns are overlapping and either reduce them or prolong the station keeping cycle to avoid the intersection (assuming that two thrusters cannot burn for the same time for example). The output of this step is the final burn location and duration.

4.2 The Hopping Manoeuver scheme

4.2.1 Inner working

The scheme that we have sought to develop is called the "hopping" scheme. The cycle consists of two sets of maneuvers that together can compensate perturbation and errors. A maneuver is defined as being 4 burns, one per each thruster. To avoid overlapping burn, we decide to make an "hop" in eccentricity and in-plane angular momentum and then "hop" back in addition of the compensation of the perturbation and error. To give an example, the first maneuver could hop of $e_x = 0.5 \times 10^{-5}$, $e_y = 0$, $h_x = 0$ Nms and $h_y = -7$ Nms, and then the second maneuver could hop back of $e_x = -0.5 \times 10^{-5}$, $e_y = 0$, $h_x = 0$ Nms and $h_y = 7$ Nms. One could formulate the change in eccentricity and angular momentum in a more general way by writing $\Delta_{M1} = \Delta/2 + \Delta_{hop}$, where Δ_{M1} is change induced by the first maneuver, and $\Delta_{M2} = \Delta/2 - \Delta_{hop}$, where Δ_{M2} is change induced by the second maneuver. Thus, the cycle of the two maneuver would acheive a change of $\Delta_{M1} + \Delta_{M2} = \Delta$.

A way of manually implementing this can be achieved by alternating the sequence of burn. For example, a typical first maneuver sequence should be firing NW then NE then SE then SW, and then switching the order to NE then NW then SW then SE. The example given can be used in real life only if combination of thruster position and direction is symmetric enough.

In general, to achieve to hop back and forth, one should search the size of the hop in eccentricity and in in-plane angular momentum. Those should be determined by either optimization scheme or trial and error by checking if the maneuver algorithms can find a set of burns capable of doing the hop in and the hope back. The hope sizes depend on the mission requirements, design of the spacecraft and size of the reaction wheel.

4.2.2 Example

Satellite settings We assume an earth pointing spacecraft with an allowed attitude bias up to a few degrees. On the earth pointing platform the thrusters are assumed to be symmetrically positioned and oriented such that the resulting thrust and torque directions are linearly independent. The configuration of fixed thrusters is illustrated in Figure 2 and Figure 3 which show the two redundant branches of four thrusters and their schematic positions along with their plume directions. The size is indicated by letters a, b, c and the (absolute) angle toward the north south axis is for all thrusters.

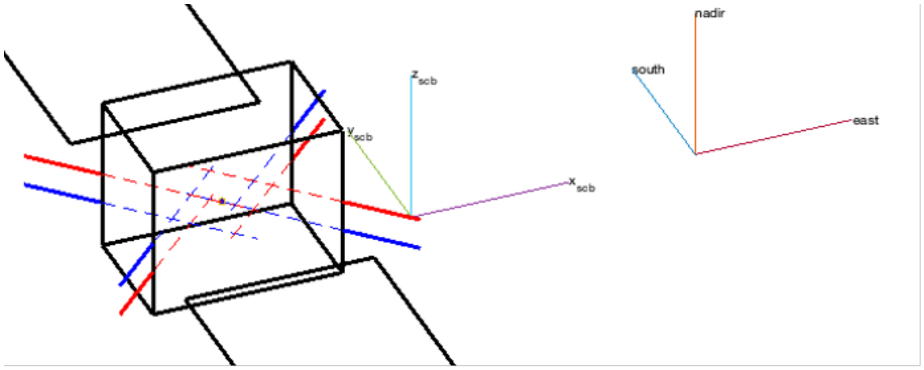

Figure 2: Schematic of the two thruster branches (in red and blue) with plume directions and the spacecraft coordinate system as well as the orientation of the spacecraft on orbit.

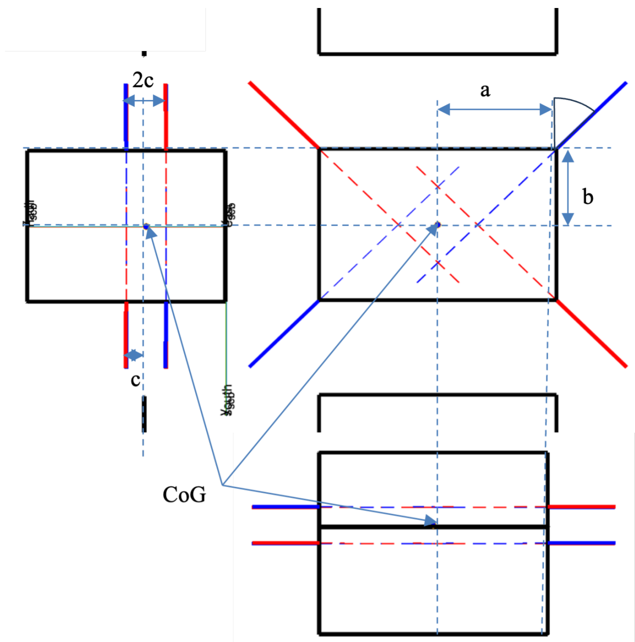

Figure 3: The spacecraft in different projections and with indications of the positions of the thrusters with dimensions indicated by, a, b and c. The centre of gravity is inside the body of the spacecraft and the thrust vectors are intentionally directed off COG

Here we will assumed that the absolute angle toward the north south axis is 45° for the north poiting thrusters, and 135° for the south thrusters, the thrust direction is assumed purely tangential (along track). We will use the following dimension : $a = 0.6$m, $b = 0.5$m, $c = 0.1$m, with a mass of 1000 kg. We will also assume no plume inpengement nor thruster missalignement. Regarding thruster performance, we assume 15mN of thrust provided by all the thrusters. Finaly the cross section of the satellite is 13 m^2

Performances A typical result of this scheme can be seen in the following pictures. The scheme uses 6 maneuvers over 14 days, so 3 cycles of a pair of hops. The maneuvering days occur almost every other day, leaving the Sundays for orbit determination. In the figure 4 we can see the alternating sequence of the burn.

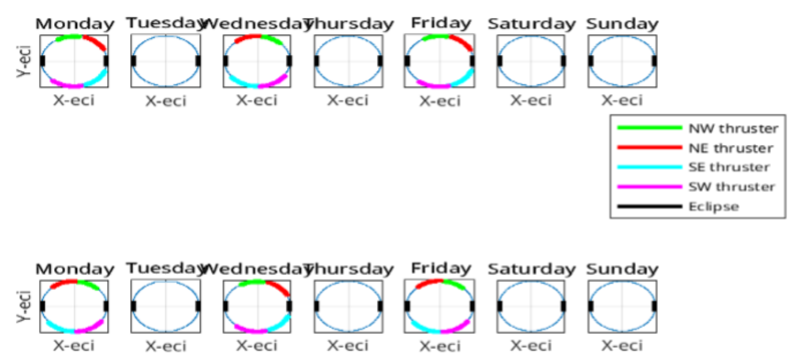

Figure 4: Maneuvers per day during 14 days

To better visualize the "hops" we can look at the Eccentricity plot in Figure 5. We can see two movements starting from $(0, 0)$ and going to $(-1 \times 10^{-4}, 0)$, via the bottom arcs, and then we hope back via the top arcs. This figure also provides proof that the burns are not intersecting, since each color is an arc from a circle. If they were two thrusters burning at the same time, one would see a straight evolution instead.

Since, we are hoping in the angular momentum plane, we also see the back and forth in Figure 6, where the plateau value are at different level after each maneuver for the X and Y component. The Z component is only a straight line since the simulation is modeling for an angular momentum perturbation on that axis.

This technic leads to the following performance in term of position over time seen in Figure 7 and position withing the geostationary box in Figure 8. With both figure, we see that we are well within the box, with little to no build up over the 14 days period.

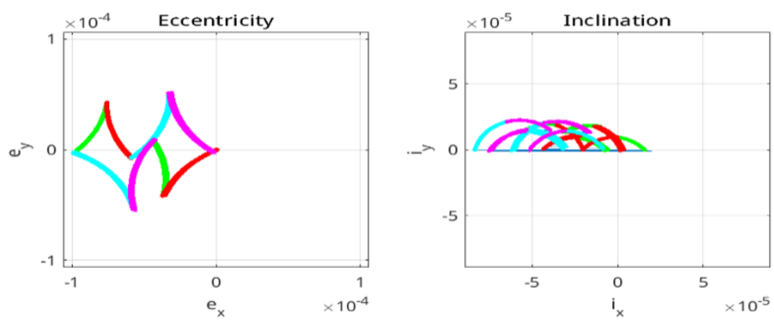

Figure 5: Evolution of Eccentricity and Inclination over time

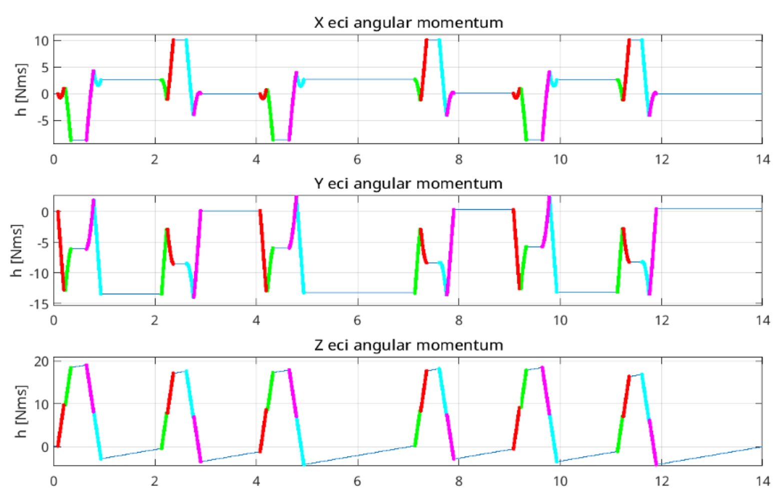

Figure 6: Angular momentum over time

4.3 Manoeuver Determination algorithm

At the heart of our Algorithm is our new way of computing manoeuver capable of achieving the desired change in the orbital element. To do that we have writen down all the equations and simplify them as much as possible before solving them numerically, via the symbolic MATLAB toolbox. A details version of which equations and how we approch to simplify them is done in the appendix,

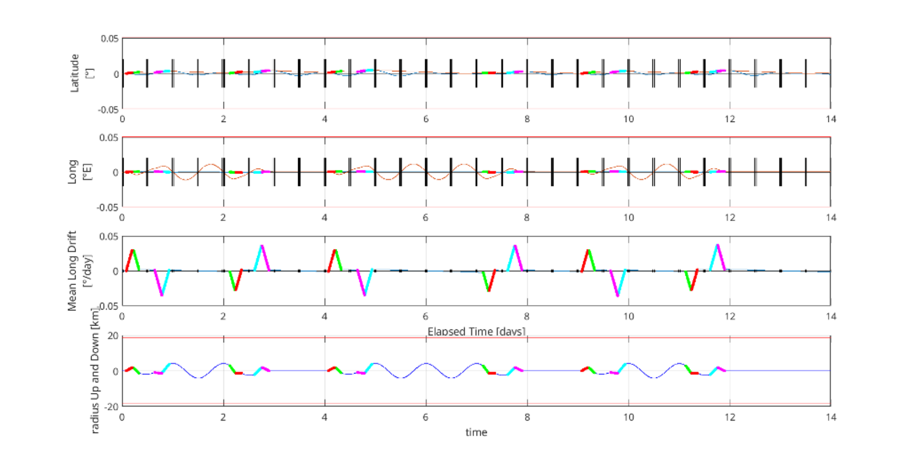

Figure 7: Position over time

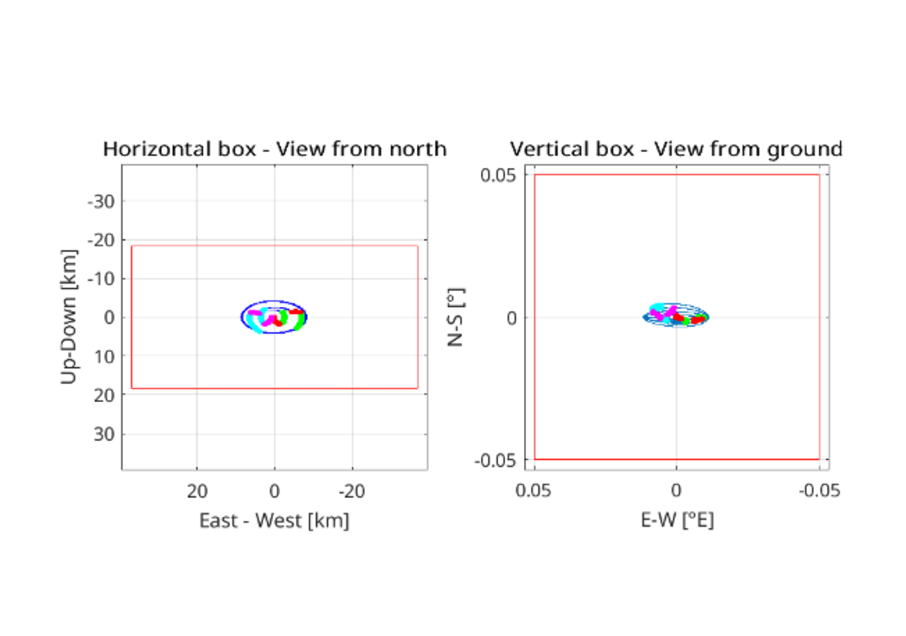

Figure 8: Position evolution in the geostationary boxes

we will give a brief overview of the algorithm here in this subsection.

The first step is to derived the equations linking for each thruster burn to the variation of each state variable. One will obtain two sets of equation a time dependent one or depending of the sideral angle of the burn and the other one time independent. Then, one need to invert the systems of equations since one is interested to find the thrust parameters in function of the desired changes in the state variables (orbital element and angular momentum). One can to that by first using a Gauss elimination process to simply the two sets of equations. Then, we can combine the two sets of equations to obtain two polynomial equations depending of the information of one thruster burn (duration of the burn and sideral angle of the burn).

The second step is solving the equation numerically. The two polynomial equations are then solve using the Groebner algorithm [2] to simplify the polynomial system and then use MATLAB symbolic toolbox to solve it.

The third step is to generated all information about all burns by reinjecting the solution into the equations and solving the system to get the set of the four burns.

It is to be noted that the design of the satellite has a direct effect on the ease of solving the equation. Major simplifications can occur from general equations by adding symmetry in thruster positions or directions. A difference could be seen as the different set of equation presented in the Appendix.

5 Consideration

5.1 Sizing

The sizing of the system is a critical aspect of the design process. The propulsion system, including the reaction wheel, must be appropriately sized with respect to the following factors: the direction of the thrusters, their placement, and the thrust level generated by the thrusters.

These considerations, along with mission duration and, to some extent, the spacecraft's longitudinal position on the geostationary ring, determine the total impulse required. This, in turn, influences the amount of propellant needed, which is a function of both the specific impulse and propellant mass.

For station-keeping requirements, the primary factor is the thrust direction. The angle between the thruster direction and the North-South axis is the main driver for fuel consumption and efficiency. This angle can be influenced by factors such as plume impingement and the accommodation of the thrusters. Typically, an angle of 45° is used for electric thrust systems.

Beyond the North-South angle, the positioning of the thrusters also impacts the propellant budget and the simplicity of solving the governing equations. Thruster placement is primarily determined by the spacecraft's dimensions and the practical constraints for accommodation. Careful consideration must be given to thruster positioning and thruster direction, as it directly affects angular momentum management via the size of the moment arm, and, consequently, this will drive the angular momentum bandwidth required by the algorithm.

5.2 Flexibility

The maneuvering scheme provides a wide range of variations to accommodate mission constraints. The first parameter that can be adjusted is the size of the hop. Once the burn parameters are determined, the number of cycles per week can be varied, allowing for shorter burns, which results in slight efficiency gains and more precise control of the spacecraft's trajectory. Additionally, the number of burns performed each day can be modified. For example, the setup described above uses four burns in one day, but this could be adjusted to two burns spread over two days, while still ensuring one burn per thruster during each hop. This approach helps avoid overlapping burns, which becomes particularly useful when the thruster positions introduce a larger radial component.

6 Conclusion

This paper demonstrates how the Hopping scheme and the deterministic algorithm developed by Vinterstellar can offer a simpler alternative to spacecraft configuration. By using a fixed thruster configuration rather than a thruster pointing or orientation mechanism, our approach enables simultaneous control of both angular momentum and orbital parameters. The algorithm allows for a wide range of maneuver options, offering the flexibility to accommodate a variety of mission constraints. Whether it is performing additional cycles, extending mission operations over two days, or adjusting the spacecraft's trajectory to meet new requirements, the tuning of all the parameters of the algorithm together with the spacecraft design are determinants for a successful usage of our algorithm. Hence, by simplifying satellite design, this approach aims to transform station-keeping strategies, particularly for smallsat missions where cost mass and operational simplicity are critical.

References

[1] Michael Francis Atiyah and Ian G. Macdonald. *Introduction to commutative algebra*. eng. Addison-Wesley series in mathematics. Boca Raton London New York: CRC Press, Taylor & Francis Group, 2018. ISBN: 978-0-201-40751-8.

[2] Bruno Buchberger. "Bruno Buchberger's PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal". In: *Journal of Symbolic Computation*. Logic, Mathematics and Computer Science: Interactions in honor of Bruno Buchberger (60th birthday) 41.3 (Mar. 1, 2006), pp. 475–511. ISSN: 0747-7171. DOI: 10.1016/j.jsc.2005.09.007. URL: https://www.sciencedirect.com/science/article/pii/S0747717105001483 (visited on 05/30/2024).

[3] David A. Cox, John B. Little, and Donal O'Shea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. 3rd ed. Undergraduate texts in mathematics. New York: Springer, 2007. 551 pp. ISBN: 978-0-387-35650-1.

[4] Karen E. Smith et al. *An Invitation to Algebraic Geometry*. Universitext. New York, NY: Springer, 2000. ISBN: 978-1-4419-3195-5. DOI: 10.1007/978-1-4757-4497-2. URL: http://link.springer.com/10.1007/978-1-4757-4497-2 (visited on 05/30/2024).

[5] E.M. Soop. *Handbook of Geostationary Orbits*. 1994. ISBN: 978-90-481-4453-2. URL: https://link.springer.com/book/9789048144532 (visited on 05/23/2024).

[6] Jiri Vesely and Sang Van Doan. "Analytical method solving system of hyperbolic equations". In: *2015 25th International Conference Radioelektronika (RADIOELEKTRONIKA)*. 2015 25th International Conference Radioelektronika (RADIOELEKTRONIKA). Pardubice: IEEE, Apr. 2015, pp. 343–348. ISBN: 978-1-4799-8117-5 978-1-4799-8119-9. DOI: 10.1109/RADIOELEK.2015.7129064. URL: https://ieeexplore.ieee.org/document/7129064/ (visited on 01/09/2024).

A Thruster effects

A.1 Manoeuvre effect on orbital element

According to [5], a near geostationary spacecraft orbital element change as specify below, when we apply a change in velocity ΔV on the spacecraft at the sidereal angle s_b.

If we have a radial thrust:

$$\Delta e = \frac{\Delta V_r}{V} \begin{pmatrix} sin(s_b) \\ -cos(s_b) \end{pmatrix} \tag{9}$$

If we have a tangential thrust:

$$\Delta e = \frac{2\Delta V_t}{V} \begin{pmatrix} cos(s_b) \\ sin(s_b) \end{pmatrix} \tag{10}$$

$$\Delta D = \frac{-3\Delta V_t}{V}. \tag{11}$$

If we have an orthogonal velocity change:

$$\Delta i = \frac{\Delta V_o}{V} \begin{pmatrix} sin(s_b) \\ -cos(s_b) \end{pmatrix} \tag{12}$$

Instead of writing the eccentricity and inclination in a vector form, we will write them as belonging to the complex plane. We will use j as our imaginary unit to distinguish with the inclination. The concept of expressing the problem in the complex plane was proposed by [5] for the inclination and expended to the full problem including angular momentum by Emil Vinterhav in his prior work on the problem.

Using the following trigonometric changes:

$$\begin{aligned} sin(s_b) - jcos(s_b) &= cos(\pi/2 - s_b) - jsin(\pi/2 - s_b) \\ &= cos(s_b - \pi/2) + jsin(s_b - \pi/2) \\ &= exp(j(s_b - \pi/2)) \\ &= -jexp(j(s_b)), \end{aligned}$$

we can express the change in inclination in equation 12 and the change in eccentricity in both equation 9 and 10 in the complex plane, thus resulting in those equations:

$$\Delta i = \frac{\Delta V_o}{V}(-j)\exp(js_b) \tag{13}$$

$$\Delta e = \frac{2\Delta V_t}{V}\exp(js_b) + \frac{\Delta V_r}{V}(-j)\exp(js_b) \tag{14}$$

$$\Delta D = \frac{-3\Delta V_t}{V}. \tag{15}$$

We can reformulate this by the following matrix equation:

$$\begin{pmatrix} \Delta i \\ \Delta e \\ \Delta D \end{pmatrix} = \frac{1}{V} \begin{pmatrix} 0 & 0 & -j\exp(js_b) \\ -j\exp(js_b) & 2\exp(js_b) & 0 \\ 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} \Delta V_r \\ \Delta V_t \\ \Delta V_o \end{pmatrix}. \tag{16}$$

Those equations are valid for any spacecraft configuration, the geometry of the spacecraft will be taken in account when expressing the change in velocity vector in function of the trust applied.

A.2 Conservation of the angular momentum

Firing a thruster changes the angular momentum of the spacecraft, except if the thrust direction is aligned with the rotational axis of the spacecraft. We will write the change in angular momentum Δh, in the ECI coordinate frame which has the following properties: it is center at the Earth center of mass and the axis are fixed with respect to the starts. This is also a inertial system since it is not accelerating. Although this is not the easiest co-ordinate system to compute the angular momentum in. For that reason, we will use a co-ordinate system that is attached to the spacecraft body frame. The body frame co-ordinate system is center on the center of gravity of the satellite, the x-axis has for direction from the Earth to the center of gravity, the y-axis is along the trajectory of the spacecraft, and the z-axis is from south to north. That way one every orbit the axis of the two coordinates system are aligned.

So, for the computation we will first compute the change in angular momentum in the satellite co-ordinate system and then transfer it to the fix Earth centred co-ordinate system via the following transfer matrix:

$$\mathcal{P} = \begin{pmatrix} cos(s) & -sin(s) & 0 \\ sin(s) & cos(s) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

As previously, s is the sidereal angle of the spacecraft that change with time. We can note that when $s = 0$ we are in the situation where our two co-ordinates system are aligned.

It is to be noted, that transferring all the vector first and then computing the change in angular momentum is equivalent to first computing the change in angular momentum and then transferring the vector to the correct co-ordinate system, since we have the following proposition:

Proposition 1. *Let $a, b \in \mathbb{R}^3$ and $\mathcal{P}$ be a rotation matrix. Then we have the following hold:*

$$\mathcal{P}(a \times b) = (\mathcal{P}a) \times (\mathcal{P}b).$$

The last remark we need to address before, detailing the general equation is that we only care about the change in the spacecraft angular momentum induce by a torque resulting from a thrust and not the orbital angular momentum. We do so because the later include inclination and longitude drift in its formulation, and we do not want to take those changes into account again.

A change in angular momentum implied by k^{th} thruster satisfies this equation:

$$\Delta h = \mathcal{P}\left(r_{G\to k} \times m\Delta V\right), \tag{18}$$

where h is the angular momentum, $r_{G\to k}$ is the vector from the center of mass of the spacecraft to the thruster k.

A.3 Relation between the thrust impulse and the change in the state space variables

For a given configuration of thruster, we have that:

$$\Delta V = \frac{1}{m}[n_k]\mathcal{T}, \tag{19}$$

where m is the mass of the spacecraft, $[n_k]$ is a matrix whose columns are the thrust direction of each thruster k, and $\mathcal{T} = \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix}$ where each T_k represent the k^{th} thruster impulse, and we have that

$$T_k = F\Delta t, \tag{20}$$

where Δt is the time duration of the burn, and F is the force of the thruster, we assume that all the thruster output the same force.

Remark: For our application it is enough to solve the problem for impulsive thrust, meaning a thrust very short like a Dirac. This can also be seen as a thrust with a high force. However, by the design of the spacecraft we have low thrust, so we will apply a low force for a longer period of time. Fortunately, there exist a way to go from the continuous thrust to the impulsive thrust by multiplication of $\frac{2\sin(s_b)}{s_b}$ where s_b is the sidereal angle corresponding to the middle of the burn, this just a multiplicative constant that arise from the integration of the formulas. Thus, from now on, we will consider only impulsive thrust.

If we put everything together, we have the following change:

$$\begin{pmatrix} \Delta i \\ \Delta e \\ \Delta D \end{pmatrix} = \frac{1}{mV}\begin{pmatrix} 0 & 0 & -j\exp(js_b) \\ -j\exp(js_b) & 2\exp(js_b) & 0 \\ 0 & -3 & 0 \end{pmatrix}[n_k]\mathcal{T} \tag{21}$$

$$\Delta h = m\mathcal{P}\left(\left(\sum_{k=1}^{n} r_{G\to k}\right) \times ([n_k]\mathcal{T})\right). \tag{22}$$

These equations represent the thrust effect on the state space. But, for practical purpose, one is interested to know how which thrusts to apply to get some specific change in the orbital element. Thus, one want to inverse the above relation. It is to be noted that just by looking at the matrix size, it is impossible to have left inverse matrix and easily solve the problem, but we will see in the following section that one can still determine the thrust impulses via some geometric computation.

B Example on a simple design

B.1 Writing down the equations

B.1.1 State space variable in function of thrust

For the configuration of 4 thrusters in NW, NE, SE, SW position we have that:

$$\begin{aligned} T_{NW} &= T_{NW} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = T_{NW} n_{NW}, \qquad & T_{NE} &= T_{NE} \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = T_{NE} n_{NE} \\ T_{SE} &= T_{SE} \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = T_{SE} n_{SE}, \qquad & T_{SW} &= T_{SW} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = T_{SW} n_{SW}. \end{aligned}$$

Thus, we have that

$$\Delta V = \frac{1}{m\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \mathcal{T}. \tag{23}$$

Then combining with the equation 16, we have that:

$$\begin{aligned} \begin{pmatrix} \Delta i \\ \Delta e \\ \Delta D \end{pmatrix} = \frac{1}{mV} &\begin{pmatrix} 0 & 0 & -j\exp(js_b) \\ -j\exp(js_b) & 2\exp(js_b) & 0 \\ 0 & -3 & 0 \end{pmatrix} \\ &\times \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \mathcal{T}, \end{aligned} \tag{24}$$

which is by simplification equal to:

$$\begin{pmatrix} \Delta i \\ \Delta e \\ \Delta D \end{pmatrix} = \frac{1}{mV\sqrt{2}} \begin{pmatrix} j\exp(js_b) & j\exp(js_b) & -j\exp(js_b) & -j\exp(js_b) \\ 2\exp(js_b) & -2\exp(js_b) & -2\exp(js_b) & 2\exp(js_b) \\ -3 & 3 & 3 & -3 \end{pmatrix} \mathcal{T}. \tag{25}$$

B.1.2 Expression of the change in angular momentum

We have the following position of the Thruster:

$$r_{NW} = \begin{pmatrix} c & -a & b \end{pmatrix}^T, \qquad r_{NE} = \begin{pmatrix} -c & a & b \end{pmatrix}^T$$
$$r_{SE} = \begin{pmatrix} c & a & -b \end{pmatrix}^T, \qquad r_{SW} = \begin{pmatrix} -c & -a & -b \end{pmatrix}^T.$$

Then, the North West Thruster implies this change in angular momentum:

$$\mathcal{P}(r_{G\to NW} \times n_{NW}) = \mathcal{P}\left(\begin{pmatrix} c \\ -a \\ b \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) \tag{26}$$
$$= \frac{1}{\sqrt{2}}\mathcal{P}\begin{pmatrix} a-b \\ c \\ c \end{pmatrix} \tag{27}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -c\sin(s_b) + (a-b)\cos(s_b) \\ c\cos(s_b) + (a-b)\sin(s_b) \\ c \end{pmatrix} \tag{28}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{c^2+(b-a)^2}\cos(s_b+\eta) \\ \sqrt{c^2+(b-a)^2}\sin(s_b+\eta) \\ c \end{pmatrix} \tag{29}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{c^2+(b-a)^2}\exp(js_b)\exp(j\eta) \\ c \end{pmatrix}, \tag{30}$$

where $\tan\eta = \frac{c}{a-b}$. Then, the North East Thruster implies this change in angular momentum:

$$\mathcal{P}(r_{G\to N} \times n_{NE}) = \mathcal{P}\left(\begin{pmatrix} -c \\ a \\ b \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}\right) \tag{31}$$
$$= \frac{1}{\sqrt{2}}\mathcal{P}\begin{pmatrix} -a+b \\ -c \\ c \end{pmatrix} \tag{32}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} c\sin(s_b) - (a-b)\cos(s_b) \\ -c\cos(s_b) - (a-b)\sin(s_b) \\ c \end{pmatrix} \tag{33}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{c^2+(b-a)^2}\cos(s_b+\eta) \\ -\sqrt{c^2+(b-a)^2}\sin(s_b+\eta) \\ c \end{pmatrix} \tag{34}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{c^2+(b-a)^2}\exp(js_b)\exp(j\eta) \\ c \end{pmatrix}. \tag{35}$$

Then, the South East Thruster implies this change in angular momentum:

$$\mathcal{P}(r_{G\to SE} \times n_{SE}) = \mathcal{P}\left(\begin{pmatrix} c \\ a \\ -b \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right) \tag{36}$$

$$= \frac{1}{\sqrt{2}}\mathcal{P}\begin{pmatrix} a-b \\ -c \\ -c \end{pmatrix} \tag{37}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} c\sin(s_b) + (a-b)\cos(s_b) \\ -c\cos(s_b) + (a-b)\sin(s_b) \\ -c \end{pmatrix} \tag{38}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{c^2+(b-a)^2}\cos(s_b-\eta) \\ \sqrt{c^2+(b-a)^2}\sin(s_b-\eta) \\ -c \end{pmatrix} \tag{39}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{c^2+(b-a)^2}\exp(js_b)\exp(-j\eta) \\ -c \end{pmatrix}. \tag{40}$$

Then, the South West Thruster implies this change in angular momentum:

$$\mathcal{P}(r_{G\to SW} \times n_{SW}) = \mathcal{P}\left(\begin{pmatrix} -c \\ -a \\ -b \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) \tag{41}$$

$$= \frac{1}{\sqrt{2}}\mathcal{P}\begin{pmatrix} b-a \\ c \\ -c \end{pmatrix} \tag{42}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -c\sin(s_b) - (a-b)\cos(s_b) \\ +c\cos(s_b) - (a-b)\sin(s_b) \\ -c \end{pmatrix} \tag{43}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{c^2+(b-a)^2}\cos(s_b-\eta) \\ -\sqrt{c^2+(b-a)^2}\sin(s_b-\eta) \\ -c \end{pmatrix} \tag{44}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -\sqrt{c^2+(b-a)^2}\exp(js_b)\exp(-j\eta) \\ -c \end{pmatrix}. \tag{45}$$

Putting all things together, we have this change in the angular momentum:

$$\Delta h_{in} = \frac{m\sqrt{c^2+(b-a)^2}}{\sqrt{2}}\big[\exp(js_b)\exp(j\eta)T_{NW} \tag{46}$$

$$- \exp(js_b)\exp(j\eta)T_{NE} \tag{47}$$

$$+ \exp(js_b)\exp(-j\eta)T_{SE} \tag{48}$$

$$- \exp(js_b)\exp(-j\eta)T_{SW}\big] \tag{49}$$

$$\tag{50}$$

$$\Delta h_{out} = \frac{mc}{\sqrt{2}} \left[T_{NW} + T_{NE} - T_{SE} - T_{SW}\right] \tag{51}$$

B.1.3 Expression of all the relation

Let's write, and pose $R_1 = \frac{m}{\sqrt{2}}\sqrt{c^2 + (b-a)^2}$. Then, we have:

$$\begin{pmatrix} \Delta i \\ \Delta e \\ \Delta h_{in} \end{pmatrix} = \begin{pmatrix} \frac{j}{mV\sqrt{2}} & \frac{j}{mV\sqrt{2}} & -\frac{j}{mV\sqrt{2}} & -\frac{j}{mV\sqrt{2}} \\ \frac{2}{mV\sqrt{2}} & -\frac{2}{mV\sqrt{2}} & -\frac{2}{mV\sqrt{2}} & \frac{2}{mV\sqrt{2}} \\ R_1 \exp(j\eta) & -R_1 \exp(j\eta) & R_1 \exp(-j\eta) & -R_1 \exp(-j\eta) \end{pmatrix} \tag{52}$$

$$\times \begin{pmatrix} T_1 \exp(-js_1) \\ T_2 \exp(-js_2) \\ T_3 \exp(-js_3) \\ T_4 \exp(-js_4) \end{pmatrix} \tag{53}$$

$$\begin{pmatrix} \Delta D \\ \Delta h_{out} \end{pmatrix} = \begin{pmatrix} \frac{-3}{mV\sqrt{2}} & \frac{3}{mV\sqrt{2}} & \frac{3}{mV\sqrt{2}} & \frac{-3}{mV\sqrt{2}} \\ \frac{mc}{\sqrt{2}} & \frac{mc}{\sqrt{2}} & -\frac{mc}{\sqrt{2}} & -\frac{mc}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} \tag{54}$$

B.1.4 Expressing thrust impulse in function of Change in Orbital Element.

For computation purpose, let define the following constant:

$$\alpha = \frac{j}{V\sqrt{2}} \qquad \beta = \frac{2}{V\sqrt{2}} \tag{55}$$

$$\delta = R_1 \exp(j\eta) \qquad \gamma = \frac{3}{V\sqrt{2}} \tag{56}$$

$$\epsilon = \frac{mc}{\sqrt{2}}. \tag{57}$$

Then we obtain the following matrix:

$$\begin{pmatrix} \Delta i \\ \Delta e \\ \Delta h_{in} \end{pmatrix} = \begin{pmatrix} \alpha & \alpha & -\alpha & -\alpha \\ \beta & -\beta & -\beta & \beta \\ \delta & -\delta & \bar{\delta} & -\bar{\delta} \end{pmatrix} \mathcal{T}' \tag{58}$$

$$\begin{pmatrix} \Delta D \\ \Delta h_{out} \end{pmatrix} = \begin{pmatrix} -\gamma & \gamma & \gamma & -\gamma \\ \epsilon & \epsilon & -\epsilon & -\epsilon \end{pmatrix} \mathcal{T}. \tag{59}$$

Here $\bar{\delta}$ denote the complex conjugation of δ. It is to be noted that $-\alpha = \bar{\alpha}$. We will, do the Gauss' elimination for finding easier relation to manipulate.

$$\left(\begin{array}{cccc|ccc} 1 & 1 & -1 & -1 & 1/\alpha & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 1/\beta & 0 \\ \delta & -\delta & \bar{\delta} & -\bar{\delta} & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \cdot 1/\alpha \\ \cdot 1/\beta \\ \end{array} \tag{60}$$

$$\left(\begin{array}{cccc|ccc} 1 & 1 & -1 & -1 & 1/\alpha & 0 & 0 \\ 0 & -2 & 0 & 2 & -1/\alpha & 1/\beta & 0 \\ 0 & -2\delta & \bar{\delta}+\delta & -\bar{\delta}+\delta & -\delta/\alpha & 0 & 1 \end{array}\right) \begin{array}{l} \\ L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - \delta L_1 \end{array} \tag{61}$$

$$\left(\begin{array}{cccc|ccc} 1 & 0 & -1 & 0 & \frac{1}{2\alpha} & \frac{1}{2\beta} & 0 \\ 0 & -1 & 0 & 1 & -\frac{1}{2\alpha} & \frac{1}{2\beta} & 0 \\ 0 & 0 & \bar{\delta}+\delta & -\bar{\delta}-\delta & 0 & -\frac{\delta}{\beta} & 1 \end{array}\right) \begin{array}{l} L_1 \leftarrow L_1 + 1/2L_2 \\ L_2 \leftarrow 1/2L_2 \\ L_3 \leftarrow L_3 - \delta L_2. \end{array} \tag{62}$$

And for the out of plane component, we have that:

$$\left(\begin{array}{cccc|cc} -1 & 1 & 1 & -1 & 1/\gamma & 0 \\ 1 & 1 & -1 & -1 & 0 & 1/\epsilon \end{array}\right) \begin{array}{l} \cdot 1/\gamma \\ \cdot 1/\epsilon \end{array} \tag{63}$$

$$\left(\begin{array}{cccc|cc} -1 & 1 & 1 & -1 & 1/\gamma & 0 \\ 0 & 2 & 0 & -2 & 1/\gamma & 1/\epsilon \end{array}\right) \begin{array}{l} \\ L_2 \leftarrow L_2 + L_1 \end{array} \tag{64}$$

$$\left(\begin{array}{cccc|cc} -1 & 0 & 1 & 0 & 1/(2\gamma) & -1/(2\epsilon) \\ 0 & 1 & 0 & -1 & 1/(2\gamma) & 1/(2\epsilon) \end{array}\right) \begin{array}{l} L_1 \leftarrow L_1 - L_2/2 \\ L_2 \leftarrow L_2/2. \end{array} \tag{65}$$

Thus, we have the following relation:

$$T'_{NW} - T'_{SE} = \frac{\Delta i}{2\alpha} + \frac{\Delta e}{2\beta} \tag{66}$$

$$T'_{NE} - T'_{SW} = \frac{\Delta e}{2\beta} - \frac{\Delta i}{2\alpha} \tag{67}$$

$$T'_{SE} - T'_{SW} = \frac{-\delta/\beta \Delta e + \Delta h_{in}}{\bar{\delta} + \delta} \tag{68}$$

$$-T_{NW} + T_{SE} = \frac{\Delta D}{2\gamma} - \frac{\Delta h_{out}}{2\epsilon} \tag{69}$$

$$T_{NE} - T_{SW} = \frac{\Delta D}{2\gamma} + \frac{\Delta h_{out}}{2\epsilon}, \tag{70}$$

where $T'_i = T_i \exp(js_i)$. This T'_i is an indication of when the corresponding thruster will burn via its argument, and for how long via its module. So one can fully describe a manoeuvre given the 4 T'_i. Now, one need to resolve this system of equations to find the thrust impulses.

B.2 Hyperbola method

From the equations above one could try to solve than by hand, but there is a geometrical interpretation that it is interesting. The set of equation gives us two similar set up. One can represent the equation 69 and 66 as Figure 9 shows. The same representation could be achieved by the pair of equation 70 and 67. The figure is a representation of the given constrain and what a possible solutions could look like.

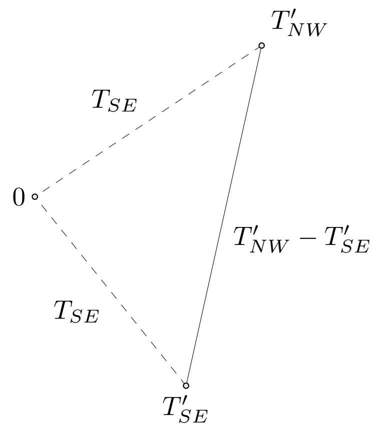

Figure 9: The dashed line represents only distance constraint, and the plain line represent constraint in both orientation and distance.

To solve the equations, one is interested of finding the value of each T'_i, but the figure 9 tells us that it is equivalent to knowing T'_i are, but not knowing where lies the origin. The equation satisfied by the origin is

$$||z - T'_{SE}| - |z - T'_{NW}|| = |\frac{\Delta D}{2\gamma} - \frac{\Delta h_{out}}{2\epsilon}|. \tag{71}$$

The meaning of this equation is searching all the possible place where the origin can lie (assuming we know where T'_{SE} and T'_{NW} lie). This equation is in fact a complex formulation of a hyperbola equation, thus there is an infinite amount of point satisfying this condition. So for solving the problem we will need the other equations.

One can do the same thing for the other pair of impulses and this lead to a new condition where 0 must be also solution of the equation:

$$||z - T'_{SW}| - |z - T'_{NE}|| = |\frac{\Delta D}{2\gamma} + \frac{\Delta h_{out}}{2\epsilon}|, \tag{72}$$

which is also a complex hyperbola.

To be able to solve the two equations above, one need to do a variable substitution in equation 71 and 72 using the last equation 68 of the system that we haven't used so far. We need to do this, since the thrust impulses are unknown, but we know the pair wise difference. Let's make $z = \hat{z} + T_{SW}$, and we obtain the following system:

$$\begin{cases} ||\hat{z} - (T'_{SE} - T'_{SW})| - |\hat{z} - (T'_{NW} - T_{SE} + T_{SE} - T'_{SW})|| = \frac{\Delta h_{out}}{2\epsilon} - \frac{\Delta D}{2\gamma} \\ ||\hat{z}| - |\hat{z} - (T'_{NE} - T'_{SW})|| = \frac{\Delta D}{2\gamma} + \frac{\Delta h_{out}}{2\epsilon}. \end{cases} \tag{73}$$

Now, we know all the constant of the two hyperbolas. If we do the substitution with the desired orbital change, we obtain the following system:

$$\begin{cases} ||\hat{z} - (\frac{\delta/\beta\Delta e + \Delta h_{in}}{\delta+\delta})| - |\hat{z} - (\frac{\Delta i}{2\alpha} + \frac{\Delta e}{2\beta} + \frac{\delta/\beta\Delta e + \Delta h_{in}}{\delta+\delta})|| = \frac{\Delta h_{out}}{2\epsilon} - \frac{\Delta D}{2\gamma} \\ ||\hat{z}| - |\hat{z} - (\frac{\Delta e}{2\beta} - \frac{\Delta i}{2\alpha})|| = \frac{\Delta D}{2\gamma} + \frac{\Delta h_{out}}{2\epsilon}. \end{cases} \tag{74}$$

The method to find the intersection of two hyperbolas is described in [6]. This method give between 0 and 4 solutions for the system. These possibles solutions correspond to possible solution for $-T_{SW}$ due to the substitution, thus leading with up to 4 quadruplet of solutions to the solution. The four possible solutions are to the intersection of the different branches of the hyperbola. So one must select the solution, if there exist one, of the two branches that we care about, these two branches are determined by the sign of $\frac{\Delta D}{2\gamma} + \frac{\Delta h_{out}}{2\epsilon}$ and $\frac{\Delta h_{out}}{2\epsilon} - \frac{\Delta D}{2\gamma}$. If there is multiple solution satisfying the above condition, one can freely pick between them. A selection criteria will be the one that lead to the minimum fuel consumption.

C Solving for every design

We have seen before in the specific example that we can solve analytically the inversion problem: meaning, for a given change in orbital element, we can compute a set of thrust that will lead to a manoeuvre that will fulfil those orbital changes. However, in the general situation one can not apply the same reasoning, mainly because the previous example was highly symmetric which lead to nice cancellation. This section is about describing a method to find a manoeuvre that could be done for a wider range of design. It is to be noted that one can still use the hyperbola method for every design, but the obtain manoeuvre won't yield the same results, the difference can be accounted for a new perturbation on the Satellite.

C.1 General Equations

The start of the reasoning is similar than in the previous case, the only difference start after the Gauss elimination set we end up in a system of this form:

$$
\begin{aligned}
T_1 - \alpha' T_4 &= \alpha && (75)\\
T_2 - \gamma' T_4 &= \gamma && (76)\\
T_3 - \beta' T_4 &= \beta && (77)\\
|T_1| - |T_3| &= \varphi && (78)\\
|T_2| - |T_4| &= \psi. && (79)\\
& && (80)
\end{aligned}
$$

If one doesn't arrive at this general system of equations when using 4 thrusters, it means that the design of the Satellite is not good. The rank of the matrix should be large enough to be able to control all the variable. In this case, the rank of the in plane matrix should be 3, and for the out of plane matrix it should be 2. Otherwise, it would mean that the thruster configuration is not able to change one state space variable and thus fail to control the spacecraft correctly during the Station Keeping phase. However, here we only have the general form of a rank 3 matrices. The out of plane matrix assume at least one axis of symmetry. One could remove this hypothesis, by considering a general form for both matrixes, but it won't be dealt here for simplicity. One could derive a solution following the same step as would be presented below.

Thus, one can reduce the problem to two equations, by simple substitution:

$$
\begin{aligned}
|\alpha + \alpha' T_4| - |\beta + \beta' T_4| &= \varphi && (81)\\
|\gamma + \gamma' T_4| - |T_4| &= \psi. && (82)
\end{aligned}
$$

The shape of the equations looks the same as the above example, the loss of symmetry is seen with the prime coefficients, which would lead to no easy cancellation to obtain a hyperbola equation.

We will focus on the equation 81, since we can retrieve easily the second one from the first.

We can rewrite the equation in that manner

$$\varphi = |\alpha + \alpha' T_4| - |\beta + \beta' T_4| \tag{83}$$

$$\sqrt{(\alpha + \alpha' T_4)(\bar{\alpha} + \bar{\alpha'} \bar{T}_4)} = \varphi + \sqrt{(\beta + \beta' T_4)(\bar{\beta} + \bar{\beta}' \bar{T}_4)} \tag{84}$$

$$\begin{aligned}(\alpha + \alpha' T_4)(\bar{\alpha} + \bar{\alpha'} \bar{T}_4) &= \varphi^2 + 2\varphi\sqrt{(\beta + \beta' T_4)(\bar{\beta} + \bar{\beta}' \bar{T}_4)} \\ &+ (\beta + \beta' T_4)(\bar{\beta} + \bar{\beta}' \bar{T}_4)\end{aligned} \tag{85}$$

$$\begin{aligned}&\left((\alpha + \alpha' T_4)(\bar{\alpha} + \bar{\alpha'} \bar{T}_4) - \varphi^2 - (\beta + \beta' T_4)(\bar{\beta} + \bar{\beta}' \bar{T}_4)\right)^2 = \\ &4\varphi^2(\beta + \beta' T_4)(\bar{\beta} + \bar{\beta}' \bar{T}_4)\end{aligned} \tag{86}$$

$$\begin{aligned}&4\varphi^2(|\beta|^2 + |\beta' T_4|^2 + 2Re(\bar{\beta}\beta' T_4)) = ((|\alpha|^2 + |\alpha' T_4|^2 + 2Re(\bar{\alpha}\alpha' T_4)) - \varphi^2 \\ &- (|\beta|^2 + |\beta' T_4|^2 + 2Re(\bar{\beta}\beta' T_4)))^2\end{aligned} \tag{87}$$

$$\begin{aligned}&4\varphi^2(|\beta|^2 + |\beta' T_4|^2 + 2Re(\bar{\beta}\beta' T_4)) = [|\alpha|^2 - \varphi^2 - |\beta|^2 + (|\alpha'|^2 - |\beta'|^2)|T_4|^2 \\ &+ 2Re((\bar{\alpha}\alpha' - \bar{\beta}\beta')T_4)]^2.\end{aligned} \tag{88}$$

Let pose for computation purposes:

$$\begin{aligned}\zeta &= |\alpha|^2 - \varphi^2 - |\beta|^2 \\ \xi &= |\alpha'|^2 - |\beta'|^2 \\ \nu &= (\bar{\alpha}\alpha' - \bar{\beta}\beta').\end{aligned}$$

So, after the substitution, we end up with:

$$\left[\zeta + \xi|T_4|^2 + 2Re(\nu T_4)\right]^2 = 4\varphi^2(|\beta|^2 + |\beta' T_4|^2 + 2Re(\bar{\beta}\beta' T_4)) \tag{89}$$

$$\begin{aligned}&4\varphi^2(|\beta|^2 + |\beta' T_4|^2 + 2Re(\bar{\beta}\beta' T_4)) = \ \zeta^2 + \xi^2|T_4|^4 + 4Re(\nu T_4)^2 + 2\zeta\xi|T_4|^2 \\ &+ 4\zeta Re(\nu T_4) + 4\xi Re(\nu T_4)T_4\end{aligned} \tag{90}$$

$$\begin{aligned}0 = &\zeta^2 - 4\varphi^2|\beta|^2 + (-4\varphi^2|\beta'|^2 + 2\zeta\xi)|T_4|^2 + \xi^2|T_4|^4 + 4Re(\nu T_4)^2 \\ &+ 4\zeta Re(\nu T_4) + 4\xi Re(\nu T_4)|T_4|^2 - 8\varphi^2 Re(\bar{\beta}\beta' T_4)\end{aligned} \tag{91}$$

Writing $T_4 = X + jY$, we can obtain the following real polynomial

$$\begin{aligned}0 = &\zeta^2 - 4\varphi^2|\beta|^2 + (-4\varphi^2|\beta'|^2 + 2\zeta\xi)(X^2 + Y^2) + \xi^2(X^2 + Y^2)^2 \\ &+ 4(Re(\nu)X - Im(\nu)Y)^2 + 4\zeta(Re(\nu)X - Im(\nu)Y) \\ &+ 4\xi(Re(\nu)X - Im(\nu)Y)(X^2 + Y^2) - 8\varphi^2 Re(\bar{\beta}\beta')X + 8\varphi^2 Im(\bar{\beta}\beta')Y\end{aligned} \tag{92}$$

$$\begin{aligned}0 = &\zeta^2 - 4\varphi^2|\beta|^2 + (4\zeta Re(\nu) - 8\varphi^2 Re(\bar{\beta}\beta'))X + (-4\zeta Im(\nu) + 8\varphi^2 Im(\bar{\beta}\beta'))Y \\ &+ (-4\varphi^2|\beta'|^2 + 2\zeta\xi)(X^2 + Y^2) + 4(Re(\nu)X - Im(\nu)Y)^2 \\ &+ 4\xi(Re(\nu)X - Im(\nu)Y)(X^2 + Y^2) + \xi^2(X^2 + Y^2)^2\end{aligned} \tag{93}$$

Thus, if we replace ζ, ξ, ν by their original value, we end up with:

$$\begin{aligned}
0 =& (|\alpha|^2 - \varphi^2 - |\beta|^2)^2 - 4\varphi^2|\beta|^2 \\
&+ (4(|\alpha|^2 - \varphi^2 - |\beta|^2)Re(\bar{\alpha}\alpha' - \bar{\beta}\beta') - 8\varphi^2 Re(\bar{\beta}\beta'))X \\
&+ (-4(|\alpha|^2 - \varphi^2 - |\beta|^2)Im(\bar{\alpha}\alpha' - \bar{\beta}\beta') + 8\varphi^2 Im(\bar{\beta}\beta'))Y \\
&+ (-4\varphi^2|\beta'|^2 + 2(|\alpha|^2 - \varphi^2 - |\beta|^2)(|\alpha'|^2 - |\beta'|^2))(X^2 + Y^2) \\
&+ 4(Re(\bar{\alpha}\alpha' - \bar{\beta}\beta')X - Im(\bar{\alpha}\alpha' - \bar{\beta}\beta')Y)^2 \\
&+ 4(|\alpha'|^2 - |\beta'|^2)(Re(\bar{\alpha}\alpha' - \bar{\beta}\beta')X - Im(\bar{\alpha}\alpha' - \bar{\beta}\beta')Y)(X^2 + Y^2) \\
&+ (|\alpha'|^2 - |\beta'|^2)^2(X^2 + Y^2)^2.
\end{aligned} \tag{94}$$

By symmetry, we can have the same other polynomial written as

$$\begin{aligned}
0 =& (|\gamma|^2 - \psi^2)^2 \\
&+ (4(|\gamma|^2 - \psi^2)Re(\bar{\gamma}\gamma'))X \\
&+ (-4(|\gamma|^2 - \psi^2)Im(\bar{\gamma}\gamma'))Y \\
&+ (-4\psi^2 + 2(|\gamma|^2 - \psi^2)(|\gamma'|^2 - 1))(X^2 + Y^2) \\
&+ 4(Re(\bar{\gamma}\gamma')X - Im(\bar{\gamma}\gamma')Y)^2 \\
&+ 4(|\gamma'|^2 - 1)(Re(\bar{\gamma}\gamma')X - Im(\bar{\gamma}\gamma')Y)(X^2 + Y^2) \\
&+ (|\gamma'|^2 - 1)^2(X^2 + Y^2)^2.
\end{aligned} \tag{95}$$

We will refer to this polynomial as P_1 and P_2 as we go on. Now, we want to find the all common root of those two polynomials, or in other word the intersection of the loci of these two polynomials. We can also note that we obtain the same equations as the previous section, if we set $\alpha' = \beta' = 1$, in this case we can easily see for example that the leading term will vanish.

C.2 Gröbner Basis

In this part, I will go a little into the detail on how one can solve an intersection of polynomials. This part will use advance mathematics in the Algebraic Geometry filed, and I will not explain all the theory behind it, if one want to learn the topic you can read the following books [3] [4] [1]. I will explain how it is working and the general idea of it.

What we want is to find the intersection of the 0 loci of polynomials, one can reformulate this into finding the variety generated by the two polynomials P_1 and P_2, one denote it $V(P_1, P_2)$. To find this variety, it is equivalent to find the variety from the ideal generate by the two polynomials. And by proposition 9 in section 2.5 of [3], if we have a basis of an ideal, say $(G_0, \cdots G_k)$, we can compute the variety $V(P_1, P_2)$ by computing the variety generated by the basis $V(G_0, \cdots, G_k)$ since, the two varieties are the same. The hope is that solving the problem with the polynomials' basis would be easier than solving it directly with the original polynomials.

The basis that we will consider is the Gröbner basis. It can be obtained of any ideal in a polynomial ring, this is provided by the Hilbert basis Theorem [3]. And this basis will provide us an easier way to solve the system of equations. One can obtain the decomposition of an ideal by applying the Buchberger's Algorithm [2].

I will consider an example of using the Gröbner basis, to better show how this can be employed for our cases.

Example: Consider the ring $k[x, y, z]$ with lexicographic ordering and let $I = (f_1, f_2, f_3) = (x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1)$ be an ideal. When applying the Buchberger Algorithm, one find the following basis:

$$g_1 = x + y + z^2 - 1 \tag{96}$$

$$g_2 = y^2 - y - z^2 + z \tag{97}$$

$$g_3 = 2yz^2 + z^4 - z^2 \tag{98}$$

$$g_4 = z^6 - 4z^4 + 4z^3 - z^2. \tag{99}$$

One can see that the last polynomial g_4 contain only one variable, so we can solve it and obtain all possible value for z. Then, we can replace z by the root found and solve g_2 and g_3 to find the possible value of y. Finally, we can solve g_1 to determine the value of x. At each step, we have solved a polynomial of only variable, which is simpler than solving a polynomial of more than two variables.

On this example, we can see that it is easier to solve the system of equation given by the g_i than the f_i because one could start by solving a one variable polynomial, then use the roots found to obtain other one variable polynomials, and then repeat the procedure. This behaviour can be done in great generality thanks to the Elimination theory, and how the Gröbner basis is computed. In short, this theory proves that the obtain basis will have the same pattern in every case to allow us to save the original system of equation. This can be seen as a generalization of the Gauss elimination.

Thanks to that technics we can find the solution of the system of equations determined by P_1 and P_2 and thus allowing us to compute the correct manoeuvre in every well-designed spacecraft. This as been implemented with the symbolic maths toolbox of Matlab and use with great success.